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Influences on High School Students' Mathematical Problem Solving Performance

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In this work we investigated the problem solving behaviors of 3 high school students as each solved three common non-routine problems with the goal to trace patterns of behaviors and performance consistency across different subject areas and problem types. Additionally, we aimed to identify possible factors that influenced children's choices of heuristics in different problem contexts. The results suggested the individual's confidence and preference for the use of certain strategies greatly impacted their mathematical problem solving practices. Inconsistency in the same individual's mathematics problem solving behaviors across different subject areas was revealed.

Key words: mathematical problem solving, high school, non-routine problems, factors, strategies.

The development of problem-solving ability among school children has been a persistent goal of mathematics education community for over a century; however, the issue of how learners may be assisted to become better problem solvers continues to be a major dilemma (Schoenfeld, 2013). This is, in part, due to the absence of specific knowledge about mathematical problem solving practices of learners and factors that influence their choices and actions (English, 2010). Indeed, previous research studies on problem solving have primarily focused on effective implementation of problem solving instruction by examining students' problem solving performance on tasks (Anderson & White, 2004). These studies have identified some key factors for the success or failure of implementation of problem solving approaches in mathematics teaching. This body of work however does not provide detailed accounts of individuals' problem solving behaviors or analysis of their affordances in the process. Muir, Beswick, and Willamson (2008) suggested that researchers must focus on understanding what successful problem solvers do and use that knowledge to help individuals develop their problem solving skills. They further argued that instead of focusing on whether particular strategies should be taught or not and how, greater attention must be devoted to understanding processes that individuals use when engaged in problem solving. In support of this suggestion we argue that knowledge about children's problem solving behaviors and factors that influence their mathematical practices while solving problems can better assist teachers in helping them nurture mature problem solvers. Such knowledge is currently not well developed. The goal of our research was to address this need.

The purpose of this study was to examine mathematical problem solving practices of three high school students in an attempt to determine whether the individuals' performances were consistent across different subject areas and problem types. Moreover, we were interested in identifying those factors that influenced the subjects' choices of heuristics they used in different problem contexts. Lastly, we intended to identify common and unique behaviors that they exhibited along with factors that seemingly motivated those behaviors.

Mathematical Problem Solving

Mathematical problem solving has been characterized as "what one does when one does not know what to do", "thinking critically about something that needs solving," "searching for best solution," "working on problems that are complex," "working on ill-defined, open-ended, and real-world problems," to list a few. These descriptions, collectively, suggest that problem solving is an activity during which the problem solver aims to find an appropriate way to cross a gap from a problem to a solution space (Flower & Hayes, 1981). Certainly, whether a task is perceived as a problem depends largely on individuals' background knowledge and their experience with the type of task under study. What might be assumed to be a problem for one person could easily be perceived as an exercise by another. In response to this issue Kilpatrick distinguished a problem as a "situation in which a goal is to be attained and a direct route to the goal is blocked" (Kilpatrick, 1985, p.2). Lesh and Zawojewski (2007) argued that a task becomes a problem when the problem solver needs to develop a more productive way of thinking about the given situation (p. 782). This description implies that the problem solver acknowledges the presence of an obstacle, recognizes the need to seek alternative approaches for solving the problem, and consents to do so.

Mayer (1985) described problem solving as "a series of mental operations that are directed toward some goal" (p. 124). Therefore, the problem solving process could be a representation of an individual's own internal exploration towards an unknown path, instead of one's ability to directly retrieve known techniques. Fully supporting Mayer's description, elsewhere we have distinguished mathematical problem solving as an activity that relies heavily on the problem solvers' in-the-moment decision making and improvising and the type of insights that they may develop in the course of their actions (Manouchehri & Zhang, 2013). We propose that the inability to develop such instantaneous insights which can assist in problem solving is not sufficient evidence to characterize an individual as ineffective or naïve problem solver since as history of mathematics has shown repeatedly in genuine mathematical problems such connections and insights may not occur immediately. As such, the activity of problem solving involves a range of complex cognitive and metacognitive actions. These cognitive processes are neither linear nor always observable. Additionally, while the problem solver may have a large (or small) amount of mathematical tools at their disposal they may not necessarily access those tools at the time when snapshots of their problem solving performance are documented. Due to this, in our work we refrain from making inferences regarding the impact of individuals' knowledge of heuristics and mathematical concepts on their problem solving performance unless there is sufficient evidence supporting such claims. Though the problem solvers in our work were not always successful in reaching solutions in places where they were asked specifically if a particular tool or concept could be used, they showed full understanding of the concept and managed to use it to attack tasks.

Effective Mathematical Problem Solving Behaviors

Nearly two decades ago, Lester (1994) summarized the research community's perspectives on qualities that distinguish successful problem solvers from those characterized as poor problem solvers, and concluded that good problem solvers: know more and their knowledge is well connected and composed of rich schemata, focus more on structural features instead of literal features of problems, are more aware of their own strengths and weakness in terms of problem solving, monitor and regulate their problem-solving efforts more routinely and are more concerned about obtaining best solutions to problems.

More recently however, English and Sriraman (2010) argued for a reconsideration of this list, indicating that since previous research had focused mainly on solving word problems typically covered in school textbooks, consisting of primarily of routine and procedural tasks, the results concerning quality of problem solving and nature of problem solving performance of children should be more critically examined. The authors attributed the lack of success of school based practices for fostering problem solving skills among children to community's inadequate knowledge about *how* individuals come to make decisions about when, where, and how to use heuristics and strategies when faced with novel problem contexts. Focusing on applying these strategies, without understanding how and why individuals make decisions about pathways for solving problems is non-productive (English, Lesh, & Fennewald, 2008; English & Siraraman 2010). Endorsing these criticisms, we propose three specific areas that merit extended inquiry if a theory of mathematical problem solving is to be developed.

Problem solving heuristics. Knowledge of heuristics and their appropriate use have been recognized as fundamental to mathematical problem solving (Schoenfeld, 1992). The study of problem solving heuristics commenced upon the Polya's work, *How to Solve it* (1945). The types of heuristics identified by Polya included: analogy, auxiliary elements, decomposing and recombining, induction, specialization, variation, and working backwards. There is evidence indicating that students' use of heuristic strategies is positively related to success in problem solving, although the effect may not always be significant (Kantowski, 1977). Yet a number of studies have highlighted the deficiencies that students exhibit when applying heuristics during the problem solving process (Schoenfeld, 1992). It remains unclear however, what factors may have contributed to the problem solvers' selection of strategies. While it is plausible to assume that the choice of strategy may have been guided by the problem solvers' knowledge of various heuristics past research has not yet provided a rational for choice or elements that may have activated the use of certain choices.

Flexibility in strategy use. Flexibility in strategy use has also been referenced as a key aspect of successful problem solving. Flexibility refers to the quantity of variations that can be introduced by an individual in the concepts and mental operations one already possess (Demetriou, 2004). Elia, Heuvel-Panhuizen, and Kolovou (2009) discussed two methods for studying strategy flexibility usage: inter-task flexibility (changing strategies across problems) and intra-task flexibility (changing strategies within problems). They used three non-routine problems to study the strategy use and strategy flexibility by 4th grade high achievers. An implicative statistical method was performed to determine whether the strategies used by students to solve the three problems were successful or not. Guess-and-check strategy was found to be the most crucial strategy that led to the success of the three pattern/algebra problems. An important finding was that more successful problem solvers exhibited higher inter-task strategy flexibility while intra-task strategy flexibility did not support the problem solvers in reaching a correct answer. An intra-task strategy flexibility study showed that the understanding of the problem influenced the correctness of the answer, instead of the flexibility of the strategies. We caution that although knowledge about how to select, flexibly from the collection those tools most appropriate to the context under study becomes a pivotal part of the problem solving, the selection process is neither straightforward nor clear. Indeed, the community's limited understanding of this complex issue continues to serve a barrier to the development of a theory of mathematical problem solving (Schoenfeld, 2013). Little research has been conducted to shed light on this matter or to verify whether similar results are observed among different populations.

Consistency of problem solving behaviors. Muir, Beswick, and Williamson (2008) studied mathematical problem solving behaviors of four 6th graders. The authors examined the strategies students used when solving 6 problems. Based on their results they identified three categories of performance: naive, routine and sophisticated. The consistency of approaches across problems for each individual was also studied, and the conclusion was that most individuals consistently exhibited behaviors characteristic in one category. Since all 6 problems used in this research concerned number and number sense, the consistency in performance across different content areas was not revealed. Absence of similar studies from the literature raises the need for a careful consideration of the conclusions of Muir and colleagues' study.

Our goal in the current study was to examine these three issues by carefully unpacking the problem solving practices of 3 high school students on 3 tasks selected from different content areas in order to first identify their choices of heuristics and the particular dispositions they exhibited and to then examine ways in which these choices and dispositions may have influenced their mathematical problem solving performance. In carrying out our inquiry we acknowledged that genuine problem solving activity is guided by mathematical sense making and is seldom linear or hierarchical in the manner that matches an algorithm; an individual moves back and forth within different layers of understanding of the problem, devising plan, testing and verifying solutions and launching answers. This back and forth movement may be necessary (Priere & Kieren, 1999) in developing a deeper understanding of the problem, leading to the individual's engagement in multiple cycles of sense making. We also recognize the presence of a multitude of internal or external forces that motivate and provoke the decisions that the problem solvers make as they attack problems and launch solutions. Some of these motives/stimuli may be initiated by the individual and some by outside information that can either facilitate or prevent making progress towards a more general understanding and ultimately, more efficient problem solving performance. Identifying these forces can be crucial to better understanding how choices are made and problem approaches are revised and refined. Therefore, we planned to document specific internal (self-initiated) and external (interviewer-initiated) forces influencing mathematical work of the participants.

Methodology

A task-based interview methodology was used to closely observe and study three students as they worked on mathematical tasks. A case study report was developed for each of the participants in which a detailed description of their actions during each interview session was recorded. These case study reports were used, first, to identify and analyze the processes that the students used and patterns of problem solving behaviors they exhibited while working on different problems; second, to describe and analyze the problem solving strategies and representations they chose to use along with factors that seemingly influenced their decisions.

Participants

This paper is a part of much larger, longitudinal research project in which we traced the development of mathematical thinking of 80 students as they progressed from 8th to 10th grade in their respective academic setting which included 12 different middle and high schools. The goal of the larger study was to study the students' modes of reasoning and their mathematical problem solving heuristic usage as they took more sophisticated mathematics courses. As such, each individual was interviewed at least once per year for a period of three years. The purpose of our research was not to trace the kind of learning that the participants may have gained from the task based interviews or to document the relationship between their knowledge of concepts and mathematical problem solving practices but to study their choices of heuristics and ways in which they navigated problems. Three participants, Jazzy in 8th grade, Liza and Yoni in 9th grade (all pseudonyms), were selected to serve as case subjects for analysis used in this report. This selection was due to important considerations as described below.

First, all three participants agreed to sign consent forms allowing their work to be used in for this project. Second, all three had worked on the same three tasks used as data collection sources. This would allow us to compare and contrast their mathematical practices, processes they used during problem solving episodes, and metacognitive behaviors they exhibited in search of common and unique patterns of performance. Additionally, the three participants offered a wide range of backgrounds and habits that would strengthen the potential for generalizability of the results. Despite their differences, the participants shared similar attributes; they were characterized as "successful" students of mathematics as measured by grades they had secured in their mathematics courses and their scores on the state standardized examinations. Lastly, the participants varied according to their claimed level of confidence in their mathematical ability and their appreciation for the subject. Jazzy felt indifferent towards mathematics and believed she was "ok in math." Yoni both liked and felt confident in his mathematical ability. Liza neither liked mathematics nor felt confident in her skills as a mathematician. Since one of our research goals was to examine common and unique patterns of mathematical problem solving practices, such diversity deemed necessary to our analysis of the problem solvers' choices and decisions. The variation in the grade levels in which the participants were enrolled was not relevant as they had taken or were taking similar courses at the time that the interviews took place.

Data sources

The data sources consisted of two individual interviews with each of the participants. Each interview lasted approximately 35-40 minutes. The participants were not restricted by time. They were also allowed to use any tools they felt they needed (manipulative materials, calculators, graph paper, etc.). Each interview was videotaped. Artifacts produced during each problem solving episode and transcriptions of the sessions were used in analysis.

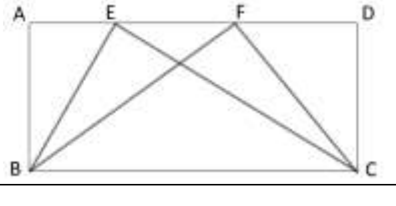
The protocol for problem solving interviews followed Goldin's guidelines (1997), mandating the least interruption from interviewers. Accordingly, interventions were made only when participants failed to explain their reasoning or their explanations were not fully understood. Scaffolding questions were allowed to be used when the participants appeared to consider problems as enigma or demanded additional information.

Data Collection Instrument

Three problems were used in the study, which concerned patterns, functions, and geometry (See Table 1 for three of the problems discussed in this paper). The diversity of subject areas and heuristics served the aim of studying the consistency of individual problem solving behaviors / performances across problem types and subject matter contexts. Each of the problems was selected to address three main goals. First, a problem where one relied on the heuristic of working backwards. In previous studies that aimed to capture the problem solvers' use of the heuristic of working backwards fractions were used. We did not wish to interfere problem solving with procedural competency with fractions. Second, as a means to study the participants' regulating actions and strategies they used we desired to include a question that challenged commonly used wrong approach. Hence, the Water Lily problem was selected as it provided for the multiple strategies problem solvers could use, potentially encouraging a shift in reasoning, assuming it was realized. Third, as we had wished to examine how the participants made decisions regarding efficiency and accuracy of their work, there was the need to include a question that provided data on participants' ability to navigate tasks that involved generalized cases. The last problem was used to address this goal. Table 1

Description of Problems

1. (Number Concepts) Joe gives Nick and Tom as much money as each already has. Then Nick gives Joe and Tom as much money as each of them then has. If at the end each has 8 dollars, how much money did each have at the beginning?
2. (Combined models) Water Lilies are growing on a lake. The water lilies grow rapidly, so that the amount of water surface covered by lilies doubles every 24 hours. On the first day of the summer, there was just one water lily. On the 90th day of summer, the lake was entirely covered. On what day was the lake half covered?
3. (Geometry) Consider the graph below: What can we say about the areas of triangles BEC and BFC?



Data Analysis

Data analysis consisted of three stages. First, for each episode we constructed a summary of the major cognitive and metacognitive actions during the entire problem solving event as a means to provide a global account of the strategies that the participants used, shifts in their choices or actions in time and well as turns during the interactions. Cognitive and metacognitive behaviors of each participant were charted as noted in Table 2. This summary was augmented with a detailed description of the processes each participant used, strategies they employed. A total of 9 episodes were used in the analysis.

Table 2
Coding Method for Key Cognitive and Metacognitive Behaviors

Coding method	Description
initial strategy	Student used the first strategy after reading the problem (e.g. writing an equation, setting up a table of value or drawing a picture)
same strategy modified information	Student modified the strategy instead of abandoning it and switching to another one (e.g. changed the initial values, extended the graph).
alternative strategy	Student switched to another strategy (e.g. moved from using equations to drawing a picture)
self-initiated justification	Student's justification for an answer directly followed his/her answer/approach.
interviewer-initiated justification	Student justified an answer upon the request from the interviewer (e.g. "Can you justify/check your answer?" "Can you convince me that this is the answer?")
self-initiated reflection	Student reflected on an answer/strategy and followed a statement or action based on the reflection (e.g. revisit the problem because of misunderstanding the information, switch a strategy because "the answer doesn't make sense").
interviewer-initiated reflection	Student reflected on an answer/strategy/action/statement upon the request from the interviewer (e.g. "Why did you do this?" "What were you thinking when you said 'oh wait, I just thought something'?")
self-initiated question	Student asked a question about the problem (e.g. "Are there any numbers in this problem?")
interviewer-initiated question	Interview asked a question about the student's strategy or the problem which led to modification/alteration of the student's behaviors (e.g. "How would you determine the sizes would even out completely?" "What information would you want me to give you in order to determine how much of the area of the rectangle that the triangle is?")

At the second level of analysis (intra-task analysis), a problem-by-problem performance model was developed for each participant in preparation for conducting a cross problem performance analysis. This phase was followed by the inter-problem analysis which allowed us to seek similarities and differences among the participants' problem solving behaviors, as well as their choices when working on problems. Common and unique patterns of problem solving behaviors were abstracted from this analysis and used when reporting the results of the study.

Results

Table 3 provides an overview of particular behaviors and practices of participants as they relate to average amount of time they spent on tasks, average number of instances of self-initiated questions, average number of times they switched strategies, average number of self-initiated testing and justifying episodes.

Table 3
Overview of Participants' Particular Behaviors and Performances

	Average length of PS episode	Average number of self initiated questions	Average number of shifts in strategy usage	Average number of justifying episodes	Average number of interviewers' scaffolding questions
Liza	11'52"	0	1	3	2
Jazzy	10'49"	0.5	2	0.5	7
Yoni	8'41"	0.5	0.5	0.2	0.3

Although the three participants varied greatly in the average amount of time they spent on tasks, their performance along the self-initiated constitutive elements of the problem solving process was consistent, suggesting potential patterns of thinking and actions. The average numbers of self-initiated questions and the frequency of shifts in strategy usage had the least variety of items, indicating less diverse use of these two behaviors. The average numbers of justifying answers, which included self-initiated justification and interviewer-initiated justification, varied more than the previous two items. The average number of questions that the interviewers asked varied greatly, highlighting practices that the participants did not exhibit naturally or by choice.

The participants' ability to identify relevant from irrelevant data, either embedded in the problem or deduced as the result of their own work, was a pivotal influence on their successful problem solving as evidenced by their willingness to reflect on options or reconsider approaches. Hence, their performances across different problems according to the types of heuristic deemed most useful in launching a solution and subject areas were not always consistent.

In the following sections we will first offer a mathematical problem solving profile of each participant, their orientations and persistent patterns of performance based on inter-task analysis. Relying on data that grounded the intra-task analysis, we will then present a cross analysis of performances to identify common and unique influences on the participants' choices.

Influences on Mathematical Problem Solving Performance

The participants' particular orientation influenced how they entered the problems during the initial phase, the degree of persistence they showed in solving them, and whether they tried to access additional strategies or engaged in metacognitive actions. The one subject (Liza) with the least amount of interest in school mathematics and its content seemed most flexible in changing strategies. Her need for understanding and sense-making, as articulated during both interview sessions, may have been the primary force behind her natural desire to constantly examine the context at hand and to switch her approaches. On the other hand, the most academically successful student among the three with most sophisticated mathematical tools (Yoni), appeared least flexible in his thinking and choices. Indeed, his attempts to use procedures he had learned in school prevented him from monitoring his progress reflectively.

Liza: The sense maker. Figure 2 illustrates the general pattern of mathematical problem solving process of Liza based on her performance on all three tasks. Liza's general problem solving process had a unique feature: sense-making. Sense-making was Liza's way of self-monitoring, which was present throughout all problem solving episodes. After she reached an answer, sense-making was her premier way to justify her response. It was one of the factors which could influence her to switch of strategy. Liza believed that she could solve most problems eventually, thus she was more likely to deliberately re-enter the problem in order to gain a better understanding of the problem.

After reading each question, Liza tended to recall a particular instance or a familiar strategy she believed to be similar the task she was encountering. The connections she made were not always relevant or productive however, she based her initial understanding of the problem and hence her initial approach to solving the problem in such associations. Once she reached a solution on a task she reflected on whether the answer made sense or not. If she thought the answer or method did not make sense (to her), or she felt the strategy was not sufficiently efficient, she attempted to adjust her approach. If the answer did not make sense to her, she would try to justify it, switch to other strategies, or revisit the problem to check her understanding of what was asked.

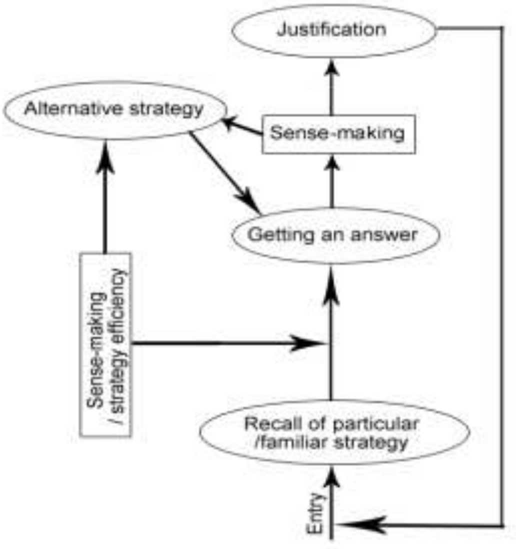


Figure 2. *Liza's general pattern of problem solving.*

The strategies that Liza chose depended largely on her understanding of the problem. If she was not successful in executing the strategy she had chosen she tended to persist using it even when she experienced difficulties. Although she showed preference for efficiency of strategies she used, she was not committed to using a specific heuristic. As such, her performance was not consistent across the content or the type of heuristic she used.

Jazzy: The number chaser. Figure 3 illustrates the general pattern of mathematical problem solving process of Jazzy based on her performance on all three tasks. Jazzy's natural choice of strategies was mostly numerical. During the entry phase, Jazzy tended to first manipulate given numbers in order to understand the problem. She relied on numerical patterns to gain a better understanding of the problem, although she was able to generalize the answer and abstract ideas following specializing in the task.

Jazzy's point of entry into all three tasks was to work with empirical data as a means to either launch an answer or increase her understanding of the problem. In all cases she exhibited the capacity to generalize the results and to abstract ideas. Due to her particular tendency to not talk or write until she was satisfied with the solution she had formulated much of her thinking was elicited through the interviewers' probing questions for either clarification or explanation.

Jazzy behaved differently depending on the type of strategy she used: if the strategy was to use numerical data and manipulating numerical information, she tended to stay with the strategy, only modified it by changing numbers. If the strategy was non-numerical, she tended to more flexibly switch to another strategy if she found her approach ineffective. She was comfortable in switching strategies if she gained a better understanding of the problem.

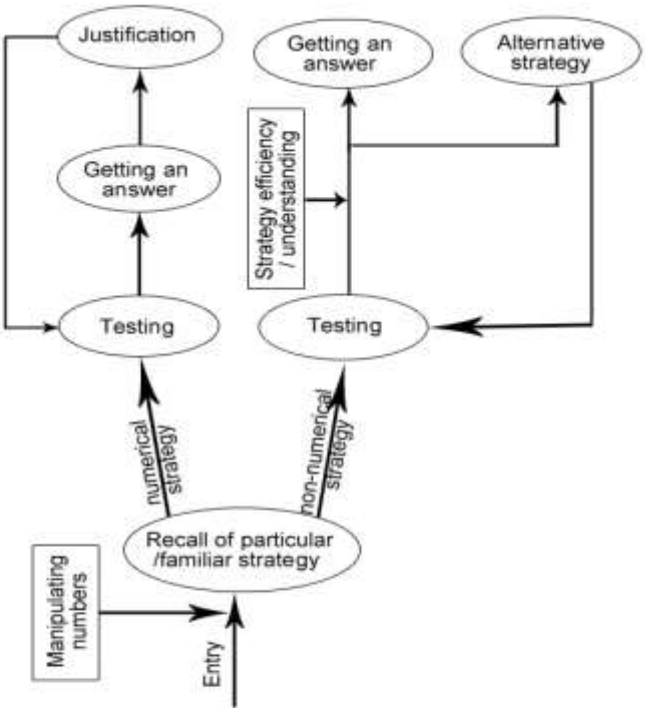


Figure 3. *Jazzy's general problem solving orientation.*

Jazzy's performance was consistent across different content areas. She routinely tried to seek and anchor her understanding of the problem in evidence derived from numerical based patterns. Using numbers as a way of specializing was a particular approach when she was not familiar with the problem. But her performance for the use of heuristics varied: she was confident and successful in her strategy usage of working backwards, but was easily confused when graphing was required. Jazzy's intra-task strategy flexibility depended on the type of the strategy she used (numerical or non-numerical). She tended to switch strategy instead of modifying information when she was not using manipulating numerical values.

Jazzy never tried to justify her answers, and did so only under the request of the interviewer. She reflected on her work when she was not confident in accuracy of her answer. When she asked questions during the problem solving episodes she inevitably elicited numerical information with which she could gain entry into the task. She rarely committed to writing on the paper unless she had mentally contemplated the task and satisfied with her idea. Long periods of silence during her sessions ranged from 1 to 3 minutes.

Yoni: The pattern gazer. Figure 4 illustrates Yoni's general problem solving pattern evidenced during the problem solving episodes.

The problem solving strategies Yoni relied on and used was guess-and-check. Throughout all problem solving episodes he constantly looked towards finding a pattern he could generalize. He was confident in appropriateness of his strategies and persisted on using them. Yoni's tendency was to stick with using one technique regardless of its success or failure in helping him launch an answer. He seldom justified his work or doubted his answers, regardless of their accuracy. His high level of confidence in his mathematical ability often prevented him from realizing the need to reflect on his work or to questions his interpretation of the problem.

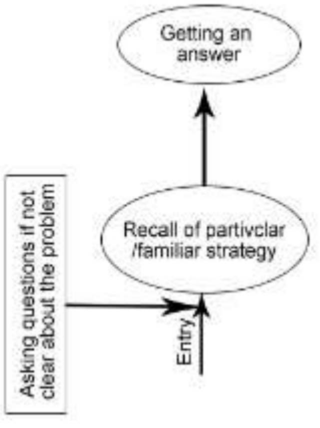


Figure 4. *Yoni's general problem solving orientation.*

As Yoni worked on a problem he asked questions if he was unsure of what he was expected to do. However, his exploration of the problem was not always influenced by the answers he received to those questions. As such once he chose a strategy to tackle the problem, he usually stayed with it until he solved the problem (unless he was asked to consider other strategies). His tendency was to look for a pattern or to find an equation that he could use to solve the problems. Although he tended to reflect on his work this reflection did not always motivate him to assess his answer or strategy even in presence of conflicting results.

Yoni's performance was consistent across different content areas and his use of heuristic. He was confident in using the heuristics of guess and check and setting up tables of values. He used these strategies regardless of their effectiveness.

Patterns of Mathematical Problem Solving Behaviors

Money transaction problem. All three participants successfully solved this problem. They relied on the heuristics of guess and check, setting up a table of values (or table-like format) as they tackled this problem. Verifying and testing accuracy of answers stapled their actions.

The initial strategy each student used was different. Liza applied a “work forward” approach as she manipulated numbers. Jazzy started the process by working backward. Yoni used guess-and-check strategy. These initial strategies revealed their desire to use the most familiar/confident strategy when coming across unfamiliar problem situations. When their initial strategy failed to help them obtain an answer, Liza changed her strategy to working backward heuristic, Jazzy and Yoni persisted on using their initial heuristic until they solved the problem.

Liza. The initial strategy Liza used is illustrated in Figure 5. She first wrote an 8 under each person's initial letter, then subtracted 4 from Joe's and Tom's final amount of money because according to her, “they each received as much of money as they got from Nick.” Then she subtracted 4 from Nick's amount of money and 2 from Tom's, explaining again that “they each had received as much money as they got from Joe.” Finally, she concluded that Joe had 4 dollars, Nick had 4 dollars, and Tom had 2 dollars at the beginning.

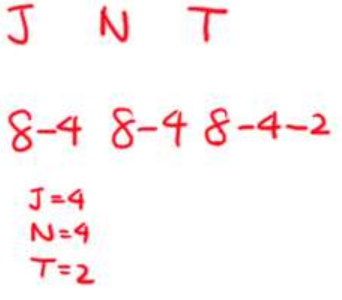


Figure 5. *Liza's initial strategy for Money Transaction problem.*

When trying to justify her answer, Liza realized Nick and Joe also had given money instead of only receiving money at each stage of transaction. She modified her strategy then (as seen in Figure 6), wrote three 8s the same as before, and then subtracted 4 from both 8s above N and T. Next she subtracted 8 from the 8 above J, and wrote the results 0, 4, 4, respectively. She explained that Nick and Tom would have 4 because “that's double 8,” and Joe would have no amount left because he gave all his away. At this step, she took the “giving money” information into consideration, but she maintained the manner she had dealt with “receiving money” in her previous strategy to deal with the “giving money” activity - subtracting a number from 8. She then reasoned that Nick could not have given any money to Joe because Joe had nothing, and wrote a 0 above the previous row of 0. She then argued that Tom would be given 4 dollars, and she added 4 to the previous 4 above T. Finally, she crossed the 4 above N and wrote 0 above it. This time she claimed that Joe had 0 dollars, Nick had 0 dollars, and Tom had 8 dollars prior to any transaction having taken place. At this step, Liza modified the way she dealt with “receiving money” activity - adding money to the original number instead of subtracting, while maintaining the “giving money” activity as subtracting. It is noticeable that in this strategy her direction of writing was in bottom to top order, which coincided with the “working backwards” heuristic. However, the motivation for this change may not have been conscious, since she was using the "working forwards" approach to justify her computation (subtracting when giving money and adding when receiving money).

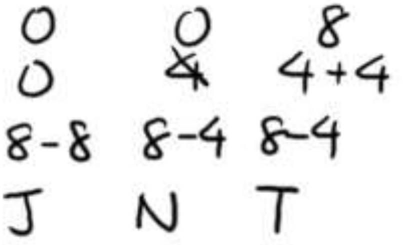


Figure 6. *Liza's modified strategy for money transaction* problem.

The alternative strategy Lisa applied following her reflection is shown in Figure 7. She started her work using a fresh piece of paper, wrote three 8s at the bottom of the paper, she then wrote the three initial letters at the very top of the paper and created a formal table, which indicated a desire to work backwards. She stated that the first step she wanted to work on was the second transaction, working backwards, and wrote two 4s above the two 8s in the columns of J and T. Repeating the computation process several times she concluded that Joe had 14 dollars, Nick had 8 dollars, and Tom had 2 dollars at the beginning. When she was asked why she seemed to be more confident with this new answer, she responded that it made more sense to her.

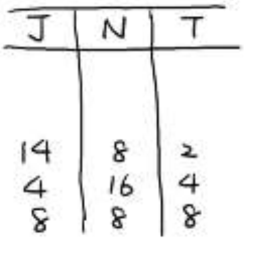


Figure 7. *Liza's alternative strategy for money transaction problem.*

Liza's initial strategy was not successful. But after she gained a better understanding of the problem, she switched to using a more effective strategy. Her new strategy was clearer and more systematic. Liza's performance on this problem was more efficient since she consciously sought to improve her understanding of the task and made informed adjustments to her approach.

Jazzy. The initial strategy Jazzy used was working backwards (illustrated in Figure 8.) At the point of entry she first wrote "Joe – 8" and "Tom - 8." Testing numbers with her calculator, approximately 1 minute after she started the problem she wrote "Nick – 32" on top. Re-reading the problem, she wrote "4" on the Tom row, and "64" on the Nick row. Her next choice was "- 76" for Joe and claimed that she had solved the problem. When she was asked to explain her answer, she claimed that she was working backwards.

She elaborated that Nick ended with 16 and he gave Joe and Tom 16, so he must have 32. She explained that if Tom was given the same amount of money he had, he should have had 4 prior to the transaction. When she was trying to justify Nick's money (64), she realized she had not obtained the correct answer.

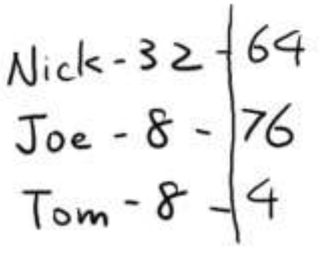
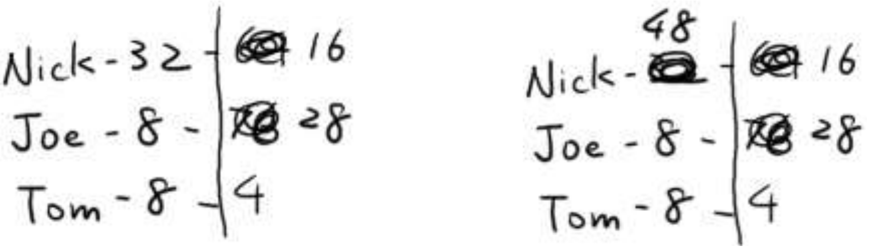


Figure 8. *Jazzy's initial strategy for money transaction problem.*

Jazzy adjusted her numbers replacing "64" with "16.” Respectively, after checking values with a calculator, she also adjusted "76" to "28" instead (Figure 9(a)). She explained, upon the request of the interviewer that she was trying to add 16, 28, and 4 to determine Nick's amount of money after the first transaction. This explanation revealed her attempts at self-regulation through numerical computation.



(a) (b)

Figure 9. *Jazzy's modified strategy for money transaction problem.*

Jazzy started a new table after these corrections and in doing so she confirmed that she had been correct before, and replaced the "48" with "32" and wrote the numbers "16," "28," "4," respectively (Figure 9(b)). Noticeably, she verified her calculations several times and despite the error she made on the second stage of transaction Jazzy's strategy was clear and effective.

Yoni. The initial strategy that Yoni used in the problem consisted of guess-and-check (working forwards), as illustrated in Figure 10.

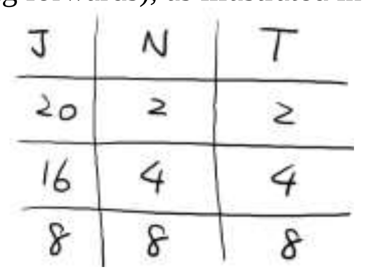
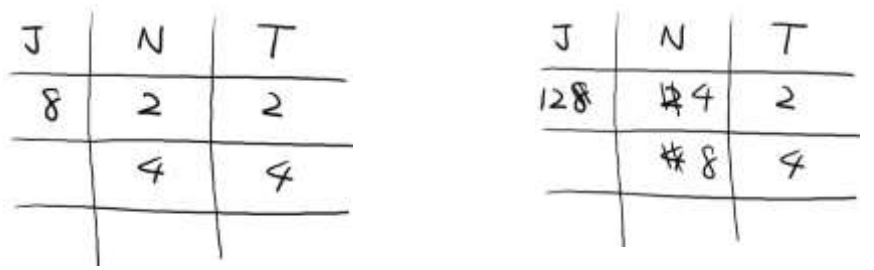


Figure 10. *Yoni's initial strategy for money transaction problem.*

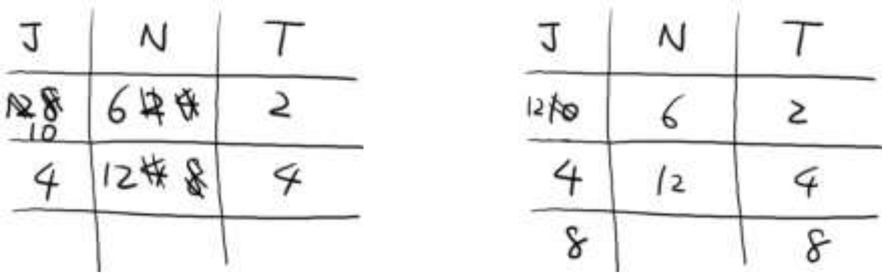
He started with the columns of N and T, filling two 2s, two 4s, and two 8s from top to bottom. Then he turned to the column of J, wrote "16 - 4 = 12 - 8 = 20," and "20 - 4 = 16 - 8 = 8." It is possible that for the "12 - 8 = 20" he was thinking about "12 + 8." He then concluded that Joe had 20 dollars at the beginning, and filled the first column with 20, 16, and 8 without providing further explanation, yet it was clear that he was subtracting the sum of two 2s from 20, and the sum of two 4s from 16 to representing that Joe gave the other two people the amount of money they already had. He felt confident in his answer and stated that the problem was solved.

He was reminded by the interviewer that Joe didn't always give money during all transactions. Yoni considered the comment and proceeded with producing new charts and attempted to adjust the values without success (see Figure 11 (a) (b) and Figure 12 (a) (b)).



(a) (b)

Figure 11. *Yoni's modified strategy for money transaction problem.*



(a) (b)

Figure 12. *Yoni's modified strategy for money transaction problem.*

Continuing with the use of guess and check for an additions of 7 minutes, Yoni managed to reach the correct answer (figure 13). Noticeably though while he insisted on using the guess and check method his approach to selection of numbers to test became more refined and deliberate.

J	N	T
14	8	2
4	16	4
8	8	8

Figure 13. *Yoni's final answer for money transaction problem.*

Yoni was never asked to justify his answer and he never attempted to do so. After quickly figuring out his first answer, the interviewer directly pointed out his misunderstanding of the information instead of asking him to justify his answer. He did not try to test his answer.

Water Lily problem. Both Jazzy's and Yoni's initial strategy was to find "half of 90." The immediate answer revealed the common conventional error we had anticipated to occur. However, their behaviors, following this initial approach differed: Jazzy immediately doubted her answer and reflected on the problem. Yoni did not doubt his answer even when his subsequent work confirmed could not confirm accuracy of it.

Liza's initial strategy was to set up a table of values, which led to the development of an insight to the problem. Her previous experience with what she considered to be a similar problem impacted her initial choice. However, her desire to find more efficient techniques to solve problems evoked her desire to abandon her initial strategy.

Liza. In this problem, Liza's entry was quick, "I remember this... with rice and I watched it on a TV show." She immediately started drawing a table of values listing in one column the number of days and the second column the number of water lilies accumulated per day, starting from one. She patently listed days 1 through 90 and computed the number of water lilies accumulated by day 13. At this point however, she said "wait a minute," pointed to the problem (around "the 90th day") and stated that the answer to the problem was 89. When she was asked whether she was sure about her answer, she studied the problem by reading it again and drew a picture to justify why she believed her answer was right. The justification is considered to be self-initiated because there was no clear request for it from the interviewer (i.e. "can you justify/check your answer" or "can you convince me that this is the answer"). The reason that she switched strategies or decided on the accuracy of her answer was not clear until she was asked to reflect on the key moment at the end of the problem solving episode - she explained that initially she was considering to use an equation instead of repetitive computation and then was inspired by the "wording" of the problem. In the problem solving process, Mason (1985) referred this as "insight," which is usually an unexpected resolution after a few calculations or years of mulling.

Liza spent only 4 minutes on the Water Lily problem, from reading the problem to the end of her final reflection. When she started drawing the table,

she seemed quite confident through her facial expression (smiling), her writing (quick), and her tone (agile). Her writing gradually became slower and finally stopped, indicating a doubt in utility of her strategy to her strategy. At the revelation of insight, she showed confidence in her answer.
In supporting her answer, Liza drew a picture circle and shaded its interior, indicating the pond was covered. She first drew the pond on the last day, which was full of water lilies, placing her pen in the middle of the pond (Figure 14) she stated that the day prior to the full coverage, the lake would have to be half full. In this manner, she was working backwards, according to her use of representation.

Figure 14. *Liza's visual representation for water Lily problem.*

If the table was the stimulator of her insight, this picture could be considered as an intuitive representation after she launched the answer. There was evidence that she did not have this picture in her mind when she declared the answer: it was only after she was asked whether she was sure about the answer that she re-read the problem, pointing at each word by the pen instead of immediately drawing the picture.

Jazzy. Jazzy paused for 30 seconds and offered that the answer was "45 days," reasoning that "it's half of 90." Only when she was asked to justify this answer, she expressed that she doubted that 45 was the correct answer. The immediate hesitation towards her first answer may be due to the absence of persuasive numerical values with which she could reason. Reflecting on the problem, she stated "I don't wanna go through every day; it takes forever," "try to think." These statements revealed an attempt to switch to a more efficient strategy, perhaps starting with setting up a table of value. Her second statement indicated that she had found the approach inefficient as she requested time to think about the problem. Following a minute long period of silence, she asked the interviewer "how many is full?" since she believed it would make it easier to solve the problem. This question could be seen as a representation of her preference for numeral sense-making (but not tedious computation). When she was told to consider full to be 1000 square feet, she quickly claimed that "when there be like 80 something, it was half full," revealing her intuitive sense that the answer should be some number close to 90. She almost immediately announced that the answer was 89th day. When she was asked to explain her answer, she argued that "500 times 2 is 1000," "so it would be the day before." She then tried to draw a picture to illustrate the answer (see Figure 15), modeling the general case. She first drew the big circle, which was to represent the pond. Then, she divided it in half, and filled the left side, stating that this was the day before the lake become completely covered. This "half and then doubled" representation corresponded to her previous numerical reasoning, which was "500 times 2 is 1000. To Jazzy, visual representation was probably more like a different form of numerical representation, not a different way of thinking about the problem.

Figure 15. *Jazzy's visual representation for water Lily problem.*

Yoni. The initial strategy Yoni applied was to divide 90 days by 2. He seemed to be sure about his answer because he rephrased the statement "it would be half of 90" three times until the interviewer asked him to show his answer using a model.

Yoni first drew a rectangle as a representation of the pond and then began drawing circles in the rectangle. He started with 1 circle, explaining that on the first day there was 1 water lily, drawing a second circle under the first one. He stated that on the second day there would be 1 more. For the third day, he reasoned that you would have 2 more, and "2 times 2 is 4," so he drew 4 more circles under the previous 2. Then he moved on to the fourth day, claimed it was 12 at that time and drew 8 circles in a new column. Having found the drawing and counting inefficient, he paused, stating that "a chart would be better."

Yoni stopped drawing after the fourth iteration as the number of circles became two cumbersome to produce, and tried to set up a table of value. Instead of writing each day in a sequence, he claimed that he would "skip days" although he would "still count them." He entered in the day-column numbers 1, 5, 10, 15, 20, ..., 90. As he reached the value of number lilies iteratively, multiplying by 2 each time, he stated that he needed a pattern. He restarted the computation by mind, timing 1 by 2 for five times, and filled 32 next to the Day 5. Using this as a point of entry then he attempted to fill the table by calculating powers of 2^5 and claimed that he had "a pattern that multiplies by 32 every time." When he was asked to justify his original answer (45th day) by this table, he stated that he could go through the rest of the table in this way, and then jumped to the last row to multiply the number on the 45th day by 32 for 9 times. Since the calculator could not produce the number he desired he then resorted back to his original answer of 45 days.

Compare area problem. The initial strategy each student applied was different. Liza looked at similar/congruent triangles in the graph, claiming that she had just worked on similar problems in her geometry class. Jazzy's initial strategy was to try and compute the area of each triangle and comparing the results. She insisted that she did not have sufficient information to solve the problem because measures of the sides of the triangle were not provided. Yoni considered part of the outer areas for each triangle, and directly compared the two triangles by the two partial outer triangles. His initial strategy seemed conceptually efficient compared to the other two participants, since he concentrated more on the relationship between areas.

The scaffolding questions were generally used to provoke deeper thoughts based on their existing reasoning in order to gain a better understanding of how they thought about the problem. Liza started with a visual perspective, and she persisted on using visual clues although she switched strategy three times. She did not try to represent the problem numerically or using area formula. She even manipulated the areas by copying them onto a new sheet. This action corresponded to her preference for using concrete models when solving problems.

Both Jazzy and Yoni asked for numbers in the picture. These questions were indicative of previous experiences with such images. However, Jazzy stayed with her question and did not try to solve the problem without numbers, while Yoni did not wait for an answer from the interviewer but began solving the problem using a visual strategy. Jazzy's confidence in relevance of information she requested made her reluctant to try a new method, while Yoni chose to dismiss doubt, believing the problem was solvable.

Liza. Liza spent 14 minutes and 49 seconds on the Compare Area problem and switched among 3 alternative strategies (initiated by the interviewer's questions). In her initial approach she attempted to show congruence of the two triangles by marking equal angles and sides she believed to be equal (see Figure 16). Connecting the problem to some of the triangle congruence criteria she had recently studied in class, she stated that the two angles were the same because they were opposite angles, and claimed that the two marked sides were the same. Her follow up attempts at establishing congruence of the triangles were unsuccessful.

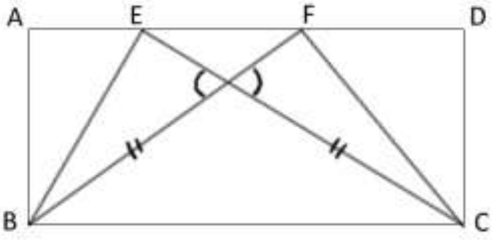


Figure 16. *Liza's initial strategy for the compare area problem.*

Following this failed approach she drew two different rectangles with each of the two triangles inscribed in one. She then attempted to compare the areas of the rectangle not included in the triangle in each case, as illustrated in Figure 17. She claimed that the shaded areas in the two models were the same as she considered pairs of triangles having the same areas (1a and 1b, 2a and 2b). The assertion of congruent may also be a connection she made to school content.

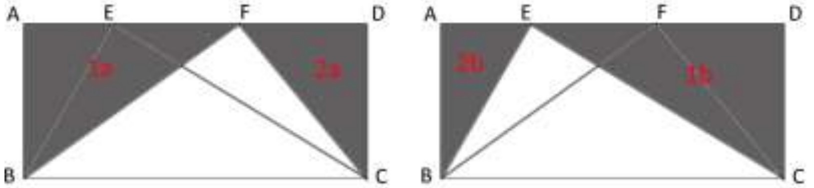


Figure 17. *Liza's alternative strategy for the compare area problem.*

In order to better determine the relationship between the sizes of the two pairs of triangles, Liza adopted another strategy, which is shown in Figure 18. She first traced the two smaller parts of shaded areas by copying them onto a new paper, and directly compared the areas. She claimed that the dark area (2a) was bigger than the light area (2b). But when she applied the same strategy to the larger shaded areas, she realized that the dark area was smaller than the light area, which contradicted to her assumption that the dark area would be always bigger than the light one. At the end of this phase, she guessed that the areas might be the same "because the sizes could even it out." Though she did not appear confident in her response.

Figure 18. *Liza's second strategy for the compare area problem.*

The last strategy Liza attempted is shown in Figure 19. She drew a grid on the rectangle and stated that she could add up the partial boxes to see how many whole boxes there were so to determine what was the relationship between the area of the rectangle and the area of the triangle.

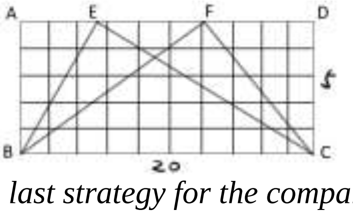


Figure 19. *Liza's last strategy for the compare area problem.*

Although Liza switched strategies 3 times, she continued to sue a visual model instead of seeking a more abstract method, that is, the triangle area formula.

Jazzy. Unlike Liza, Jazzy started the problem by asking for the measurement of sides. She refused to work on the task unless she was provided additional information. She insisted that she could not solve the problem unless measures of the sides of the rectangle.

She was asked to consider 5 and 10 as dimensions of the rectangle. Jazzy tried to use the measure to compute the ratio between the slanted sides of the triangle. This we considered as a modified strategy of the initial one, which was using proportion to compute the lengths of sides.
The interviewer asked Jazzy whether the area formulas could be used in solving the problem. This external activation lead to a quick answer on her part as she drew the heights of the triangles, stating that since the heights were the same and that triangles shared a common base then the measure of their areas were equal. Indeed, she concluded that regardless of the dimensions of the rectangle the two triangles would have areas (Figure 20).

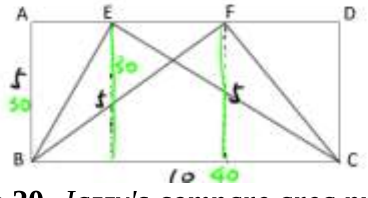


Figure 20. *Jazzy's compare area problem.*

Jazzy’s work on this problem revealed her ability in applying appropriate knowledge to solve certain problems. Provided basic knowledge, she was able to adjust it and transfer the knowledge into the current situation.

Yoni. After reading the problem Yoni asked whether there should be some numbers in the picture. Instead of waiting for an answer to his question he quickly reasoned that the area of triangle BEC would be bigger. He stated that if he could move the point E to the position of point A, then the area of triangle BEC would be half of the area of rectangle ABCD. This statement revealed that he was familiar with the relationship between the area of a rectangle and the area of a right triangle with the same base and height. He then claimed that without moving the point E, the area of rectangle BEC was a less than half of the area of rectangle ABCD because the area of triangle ABE made the triangle BEC smaller than half of the rectangle ABCD.

The initial strategy Yoni used to answer the first part of the problem is illustrated in Figure 17. He identified the two areas ABE and CDF (shaded areas in the picture), and stated that the area ABE was smaller than the area CDF. He reasoned that since the area ABE was smaller than the area CDF, the triangle BEC would be bigger than the triangle BFC.

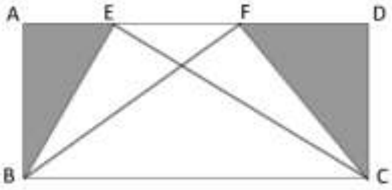


Figure 17. *Yoni's initial strategy for the first part of the compare area problem.*

Although his work was incomplete he felt confident that an answer could not be obtained since specific locations were not noted.

Discussion

The most prominent type of representation used by the three students was numerical, with preferred technique being setting up a table of values for either finding a pattern or generalizing answers. This choice was naturally used and reliance on other modes of representation occurred either as the result of the interviewers' questions or elicitation (external demand, nonetheless).

Previous studies have positively correlated self-monitoring with success in performance on certain mathematical activities (Cohors-Fressenborg, Sjuts, & Sommer, 2004; Cohors-Fressenborg, Kramer, Pundsack, Sjuts, & Sommer, 2010; Malloy & Jones 1998). Here also self-monitoring/regulating showed to be a significant influence on successful problem solving. This self-monitoring however was not always guided by a desire for efficiency or metacognitive behaviors aimed at improving work.

Consistent with findings of previous research, the results of this work suggest that intra-task strategy flexibility does not imply success at reaching correct answers to tasks (Elia, Heuvel-Panhuizen, & Kolovou, 2010). However, we posit further that the level of intra-task strategy flexibility might depend largely on the individual's confidence and preference for the use of certain strategies. These constructs may not ensure that correct answers across different subject areas and heuristic might be reached. Instead, they may prevent the individuals from moving forward in securing an enhanced level of understanding of the problem. According to the analysis of our data, high intra-task strategy flexibility could be associated with personal preference for efficient strategies, lack of confidence on a currently utilized strategy, and significant change in level of understanding of the problem. In contrast, low intra-task strategy flexibility was linked with confidence with strategy currently used and insufficient change of understanding.

The analysis of the data revealed inconsistency in the same individual's mathematics problem solving behaviors across different subject areas and/or heuristics usage. This result is distinct from the conclusion of previous research that indicates most individuals exhibit consistent problem solving behaviors (Muir, Beswick, & Williamson, 2008). In our case, the factors that may have impacted the consistency in behaviors include the preference for the use of specific approaches and orientations (visual, graphical, pictorial, etc.), experience with specific subject area (number theory, algebra, geometry), familiarity with the heuristic needed to solve the problem, and personal belief about one's own mathematical ability.

In our work it was revealed that the participants’ preference for specific subject area impacted the amount of time and energy they devoted to the problem, disregarding negative effects (i.e. unfamiliarity and frustration) he/she encounters during the problem solving process. If an individual is working on a problem that involves familiar heuristic, he/she is more likely to successfully develop an appropriate strategy to reach the correct answer even when he/she is not familiar with the problem. The preference for specific types of strategies could result in perseverance or persistence on the use of preferred strategy. Personal orientation could largely impact one's problem solving behaviors throughout the entire process (i.e. Jazzy's numerical orientation). Personal belief about one's own knowledge and ability could impact one's confidence: Liza believed she could eventually solve all problems and Yoni believed he was good at mathematics; both of them exhibited noticeable confidence during their problem solving episodes. On the other hand, Jazzy's belief about problem solving ("the only right answer") influenced her attitude during the problem solving process: she requested for the right answer even when she was convinced by visual evidence.

Based on our data we propose that the understanding of the problem is a key factor that impacts individuals' performance of solving that problem. Although this in itself is not a novel idea, we propose further that while understanding is formed at the entry phase of the problem solving process it can also be developed dynamically throughout the entire process. As such one can assume understand the problem better, it itself, as problem solving. The level of understanding could impact the choice, modification, or switch of strategies, and certain metacognitive behaviors, as well as the efficiency of these activities.

The participants' ability to retrieve different representational modes was also driven by the contexts they had most immediately experienced in school. The use of drawing a picture for illustrating the problem became only natural for two of the participants (Jazzy and Liza) when their initial attempt at using numerical data for answering questions seemed too cumbersome to be practical. Even when they were successful in use of the strategy they remained skeptical of the accuracy of their own responses. Formalizing and authenticating the final answer derived using this approach was endorsed to an outside authority, as opposed to self-conviction.

A puzzling finding is the relationship between participants' claimed level of confidence with mathematics and their problem solving performance. In virtually all past literature focused on the connections between affect and problem solving performance of children the conclusion had been drawn that confidence and success in problem solving are directly proportional; the more confident an individual in his/her mathematical ability the better performance on problem solving was witnessed. At least in two cases we encountered conflicting results. Among the three participants, evidence of metacognitive activity of self-monitoring/self-regulating solution process was least visible in Yoni's work. This was interesting since it contrasts sharply with the body of work highlighting connections between confidence and problem solving performance. Yoni was the most confident of the three participants and perhaps the one most articulate about his taste for mathematics. However, his reflections on problems, compared to others, were least self-motivated. This could be explained, at least in part, to be the result of his success with school mathematics and his skills in controlling school exercises. As he repeatedly expressed during the interviews he was able to find equations quickly and solve problems that the teacher assigned him to do. Rarely was he inclined to question validity of his approach since he was confident that his work always led to correct answers.

In contrast, Liza, least skilled in the use of school mathematics was most motivated by whether answers made sense to her or not. Indeed, it was this particular desire that allowed her to move flexibly among different representations and strategies when solving problems. The primary motive for her monitoring and regulating answers and her own actions was whether the process was internally meaningful. In places where she was unable to justify or explain accuracy of her answers she naturally folded back and considered other options.

Lastly, the need for interviewer scaffolding became crucial in assuring participants' engagement in problem solving. In all three cases, scaffolding questions posed by the interviewer served as fundamental impetus for "looking back" phase of problem solving process of the participants. While in varying degrees, all three participants' problem solving process was positively impacted by scaffolding questions. Indeed, in case of one of the subjects' reluctance to even exploring the task was prominent. It is likely that repeated exposure to procedural tasks and routine problems can have negative effect on students' ability to explore non-routine problems. Students might either limit their exploration within a "routine" bound, or try to make connections between a non-routine problem and a more familiar routine problem (Compare Area problem in this study), which is sometimes unnecessary or inefficient. Although findings of previous research on the impact of instruction on heuristic usage has been inconclusive (Schoenfeld, 2007), we argue, based on the episodes of scaffolding portions of the interviews that introducing a strategy without bounding it to a certain type of problem might be a productive venue to pursue in classroom. A potentially beneficial way to fulfill such a goal could be using various types of examples/problems for one strategy, or applying various strategies to the same problem.

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